Inequality 8

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Q> If $a, b, c > 0$ and $(1+a)(1+b)(1+c) = 8$ then prove that $abc \leq 1$

Ans:- $\frac{1+a}{2} \geq \sqrt{a} \Rightarrow 1+a \geq 2\sqrt{a}$ $\quad 1+b \geq 2\sqrt{c}$ $\quad 1+c \geq 2\sqrt{c}$

$\Rightarrow (1+a)(1+b)(1+c) \geq 8\sqrt{abc}$

$\Rightarrow abc \leq 1$

Q> For $x, y, z \in \mathbb{R}$, $x^2+y^2+z^2 \geq xy+yz+zx$

Ans:- $\frac{x^2+y^2}{2} \geq xy \Rightarrow x^2+y^2+z^2 \geq xy+yz+zx$

Q> If $a > 1$, then prove $a^n - 1 > n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)$

Ans:- $a^n - 1 \geq n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)$

$\Leftrightarrow (a-1)(a^{n-1}+a^{n-2}+\cdots+1) \geq n a^{\frac{n-1}{2}}(a-1)$

$\Leftrightarrow \underbrace{(a^{n-1}+a^{n-2}+\cdots+1) \geq n a^{\frac{n-1}{2}}}_{\text{If we can prove this problem is done}}$ $\quad [a-1>0]$

$\frac{1+a+\cdots+a^{n-1}}{n} \geq \sqrt[n]{a^{\frac{(n-1)n}{2}}} = a^{\frac{n-1}{2}}$ by AM-GM

$0+1+2+\cdots+n-1 = S$

So it is true

$S = 0+1+2+\cdots+n-1$

$S = n-1+n-2+n-3+\cdots+0$

$2S = \underbrace{n-1+n-1+n-1+\cdots+n-1}_{n \text{ times}}$

$\Rightarrow 2S = n(n-1)$

$\Rightarrow S = \frac{n(n-1)}{2}$

Q> If $0 < b \leq a$ then $\frac{1}{8}\left(\frac{(a-b)^2}{a}\right) \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8}\left(\frac{(a-b)^2}{b}\right)$

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Ans:- $\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}$

$a \geq b$

$\sqrt{a} \geq \frac{b}{\sqrt{a}}$

$\frac{1}{8}\frac{(a-b)^2}{a} = \frac{1}{8}\left(\sqrt{a} - \frac{b}{\sqrt{a}}\right)^2$

$\left(\sqrt{a} - \frac{b}{\sqrt{a}}\right) \geq 0$

$\frac{1}{8}\left(\sqrt{a} - \frac{b}{\sqrt{a}}\right)^2 \le \frac{(\sqrt{a}-\sqrt{b})^2}{2}$

$\sqrt{a} \geq \sqrt{b}$

$\sqrt{a} - \sqrt{b} \geq 0$

$\Leftrightarrow \frac{1}{4}\left(\sqrt{a} - \frac{b}{\sqrt{a}}\right)^2 \le (\sqrt{a}-\sqrt{b})^2$

$\Leftrightarrow \frac{1}{2}\left(\sqrt{a} - \frac{b}{\sqrt{a}}\right) \le (\sqrt{a}-\sqrt{b})$

$\Leftrightarrow \frac{\sqrt{a}}{2} - \frac{1}{2}\frac{b}{\sqrt{a}} \le \sqrt{a} - \sqrt{b}$

$\Leftrightarrow \frac{\sqrt{a}}{2} \geq \sqrt{b} - \frac{1}{2}\frac{b}{\sqrt{a}}$

$\Leftrightarrow \frac{a}{2} \geq \sqrt{ab} - \frac{b}{2}$

$\Leftrightarrow \frac{a}{2} + \frac{b}{2} - \sqrt{ab} \geq 0 \Leftrightarrow \frac{a+b-2\sqrt{ab}}{2} \geq 0 \Leftrightarrow \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$

Similarly upper bound.

Q> $a, b, c > 0$ then prove that, $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab+bc+ca$

Ans:- $\frac{a^3}{b} + \frac{b^3}{c} + bc \geq 3\sqrt[3]{\frac{a^3}{\cancel{b}}\frac{b^3}{\cancel{c}}\cancel{bc}} = 3ab$

$\frac{b^3}{c} + \frac{c^3}{a} + ca \geq 3bc$

$\frac{c^3}{a} + \frac{a^3}{b} + ab \geq 3ca$

$2\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) + ab+bc+ca \geq 3(ab+bc+ca)$

$\Rightarrow \frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab+bc+ca$

Q> For $a, b\ c \geq 0$ prove that

$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$

Ans:- $\frac{a^2b^2 + b^2c^2}{2} \geq \sqrt{a^2b^4c^2} = ab^2c$

Similarly for other.

So done

Rearrangement Inequality :-

$a_1 \le a_2 \le \cdots \le a_n$ be two sequences
$b_1 \le b_2 \le \cdots \le b_n$

Then for any permutation, $(a_1', a_2', \cdots, a_n')$ of $(a_1, a_2, \cdots, a_n)$ we get that,

$$\underbrace{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}_{\text{Maximal Sum}} \geq a_1' b_1 + a_2' b_2 + \cdots + a_n' b_n$$

$$\geq \underbrace{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}_{\text{Minimal Sum}}$$

Proof :- $S = a_1 b_1 + a_2 b_2 + \cdots + a_{n_1} b_{n_1} + \cdots + a_{n_2} b_{n_2} + \cdots + a_n b_n$

$S' = a_1 b_1 + a_2 b_2 + \cdots + a_{n_2} b_{n_1} + \cdots + a_{n_1} b_{n_2} + \cdots + a_n b_n$

In S' only a_{n_1} and a_{n_2} gets permuted.

$$S - S' = a_{n_1} b_{n_1} + a_{n_2} b_{n_2} - a_{n_2} b_{n_1} - a_{n_1} b_{n_2}$$

$$= a_{n_1}(b_{n_1} - b_{n_2}) + a_{n_2}(b_{n_2} - b_{n_1})$$

$$= (b_{n_1} - b_{n_2})(a_{n_1} - a_{n_2}) \geq 0$$

$a_{n_1} \le a_{n_2}$
$b_{n_1} \le b_{n_2}$

$\Rightarrow S \geq S'$

Repeating this 2-element swap we can get any permutation.
So S is maximal sum

Similarly for smallest sum

$\Rightarrow$ $a_1 \le a_2 \le \cdots \le a_n$
... be any permutation.

$\bullet >$ $a_1 \leq a_2 \leq \cdots \leq a_n$

$(a_1', a_2', \ldots, a_n')$ be any permutation.

Then $a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1 a_1' + a_2 a_2' + \cdots + a_n a_n'$

Then $\frac{1}{a_n} \leq \frac{1}{a_{n-1}} \leq \cdots \leq \frac{1}{a_1}$

$$\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + \cdots + \frac{a_n'}{a_n} \geq \frac{a_1}{a_1} + \frac{a_2}{a_2} + \cdots + \frac{a_n}{a_n} = n$$

Home Work

Q> If $a, b, c > 0$ and $abc = 1$ the prove that,

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3$$

Q> If $a, b, c > 0$ then prove that,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{9}{a+b+c}$$

Q> For real numbers $x, y > 1$ prove that,

$$\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8$$